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Theoretical confirmation of Feynman's hypothesis on the creation of circular vortices in Bose–Einstein condensates

E Infeld and A Senatorski

Sołtan Institute for Nuclear Studies, Hoża 69, 00-681 Warsaw, Poland

E-mail: einfeld@fuw.edu.pl

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Abstract
The recent creation of pure Bose–Einstein condensates in alkali metals and also of vortices supported by them has increased interest in these phenomena. In particular, changes observed in the topology of these vortices is a partially unsolved problem. Here we confirm Feynman's hypothesis on how circular vortices can be created from oppositely polarized pairs of linear vortices. This is done by following the transformation numerically. The circular vortices so obtained satisfy known constraining relations between radii and velocities.

1. Introduction

Recently Bose–Einstein condensates (BECs) have been created in alkali metals [1]. This fact has renewed interest in various phenomena observed in these media. The changes observed in the topology of superfluid helium vortices have intrigued people for some time now [2]. Now pure BEC vortices pose similar questions. These vortices either extend from wall to wall, however tangled they may be in between, or else they can be roughly circular and freely move around the superfluid [2]. Some time ago, Feynman [3] postulated that two oppositely polarized line vortices could, if they cross at two points, reconnect so as to create a circular vortex that snaps off and subsequently lives a life of its own. This is often simply postulated in numerical experiments, e.g. [2, 4]. That an opposite line vortex pair solution of the nonlinear Schrödinger equation (NLS) for a BEC is unstable has been demonstrated theoretically [5]. Reconnection at a point has been obtained numerically [6]. What remained to prove was that a known, stationary, double vortex line solution can thus reproduce a member of the one-parameter solitonic family of solutions for circular vortices as found for NLS [7]. In other words, can this reconnection really lead to a full confirmation of Feynman's hypothesis? Our first step was a similar changeover calculation for a limiting case, unfortunately such that the vortex configuration was degenerate [8, 9]. However, a surprisingly complete dynamic changeover from cylindrical to spherical symmetry of the soliton was observed in [8]. This augured well for the present effort.

Both a pure BEC, and also an imperfect Bose condensate such as He II, can be described by a single-particle wavefunction $\psi(\boldsymbol{x}, t)$ of N bosons of mass m that obeys the NLS, according to Gross, Pitaevski and Ginzburg:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + W_0\psi|\psi|^2. \tag{1}$$

Here W_0 characterizes the potential between bosons. Opposite vortex pair solutions, as well as those describing circular vortices, are known [7]. Each solution has a unique velocity perpendicular to the vortex plane. However, to answer the crucial question of whether dynamics can lead from the former to the latter kind, we must resort to numerics. All theory tells us is that the double line vortex configuration is unstable [5]. What develops is by no means obvious, as one could imagine the two vortices drawing together and annihilating. This, however, will be seen not to be the case.

Before giving the results of our simulations, we wish to point out that a preliminary idea of the problem can be gained from the linear version of equation (1), $W_0 = 0$. At a vortex, $\psi = 0$, so the cubic term not contributing locally might not be too surprising. However, the extent of global similarities with solutions to equation (1) may be more so. The preliminary results will help us appreciate just what the role of the nonlinear, $W_0 > 0$, term is in the act of reconnection.

One can easily check by substitution that equation (1), $W_0 = 0$, is solved by

$$\psi = \text{constant}\left[a^2 - x^2 + \mathrm{i}b\left(z(t) + \frac{\hbar t}{mb}\right)\right]\mathrm{e}^{\mathrm{i}(k_z z - \hbar k_z^2 t)/(2m)}, \qquad z(t) = z - \frac{\hbar k_z}{m}t, \quad b > 0. \tag{2}$$

Vortices are situated where both Re ψ and Im ψ are zero. They constitute two oppositely polarized lines along y at $x = \pm a$ and move together at velocity $U_z = \hbar(k_z - b^{-1})/m$. As we see, there is no correlation between the separation, $2a$, and the uniform velocity U_z, which can in fact have either sense.

A second solution to equation (1), $W_0 = 0$, is given by [10][1]:

$$\psi = \text{constant}\left[R^2 - x^2 - y^2 + \mathrm{i}d\left(z(t) + \frac{2\hbar}{md}t\right)\right]\mathrm{e}^{\mathrm{i}(k_z z - \hbar k_z^2 t)/(2m)}, \qquad z(t) = z - \frac{\hbar k_z}{m}t. \tag{3}$$

A circular vortex at $x^2 + y^2 = R^2$ is now moving up z at velocity $U_z = \hbar(k_z - 2d^{-1})/m$. Again, there is no connection between R and the uniform velocity, or even with its sense.

2. Known stationary solutions

Jones and Roberts [7] found both a class of stationary, double line vortex solutions to (1), $W_0 > 0$, as well as circular ones. Correlations between a, R and corresponding U_z were given in two tables. Otherwise, the similarities between their solutions with the above are at first surprising, especially if we choose the velocities in (2) and (3) such as to mimic those of Jones and Roberts. The role of the nonlinear term would then ostensibly be limited to ensuring that $|\psi|$ tend to a uniform value in the far field. However, there is a less obvious difference. Even if we perturbed (2) such that two vortices touched at two points, say by adding $a\cos(k_y y)$ to x initially, a circular vortex would not be produced at any $t > 0$.

[1] This paper inspired our solution (2).

Further calculations will be compared with the solutions of Jones and Roberts. Therefore we cast equation (1) in dimensionless form such that we can use their tables (here E is the average energy level per unit mass of a boson):

$$\psi \to \mathrm{e}^{-\mathrm{i}mEt/\hbar}\psi, \qquad \boldsymbol{x} \to \frac{\hbar}{\sqrt{2Em}}\boldsymbol{x}, \qquad t \to \frac{\hbar}{2mE}t, \tag{4}$$

so finally $\psi \to \sqrt{(mE/W_0)}\psi$. The unit of length so defined is known as the *healing length*. (Linear models will match the temporal dependence if $k_z = \sqrt{2Em}/\hbar$.) Now

$$2\mathrm{i}\frac{\partial\psi}{\partial t} = -\nabla^2\psi - \psi(1 - |\psi|^2). \tag{5}$$

If we write $\psi = \rho^{1/2}\mathrm{e}^{\mathrm{i}S}$, then ρ and $\boldsymbol{v} = \nabla S$ have a fluid interpretation. The variables ρ and $\boldsymbol{v}$ so defined satisfy the usual continuity equation, but due to the nonlinear term, the Newtonian equation has a rather strange pressure tensor [7]. This may explain the possibility of reconnection. Importantly, if we encircle a $\psi = 0$ line once, S must increase by $\pm 2n\pi$ so that ψ is single valued. This was the case for (2) and (3), where $n = 1$ (unless $a = 0$ in (2), in which case $n = 0$). The $n > 1$ case will be treated in a later paper.

3. Linear to circular vortices

At infinity, $|\psi| \to 1$ and this must be included in our initial conditions describing the pair of line vortices.

As initial condition, we took a two-vortex configuration in which the separation and velocity were taken from Jones and Roberts [7], table 2. Thus

$$\psi(t = 0) = \frac{r_1 r_2}{\sqrt{r_1^2 + b^2}\sqrt{r_2^2 + b^2}}\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}, \tag{6}$$

where

$$r_1^2 = (1 - 2U^2)(x + a)^2 + z^2, \qquad r_2^2 = (1 - 2U^2)(x - a)^2 + z^2,$$
$$\tan\theta_{1,2} = \frac{z}{\sqrt{1 - 2U^2}(a \pm x)}.$$

In spite of the scaling of x, following from the asymptotics of equation (5), [7], θ_i still increases or decreases by 2π when a vortex is encircled once. Note that $|\psi| \to 1$ in the far field. The constant b was chosen such that the subsequent velocity along z in the simulation would agree with that in the formula (we know from Fetter's and improved solutions [11] that $b \to 2$ as $a \to \infty$ and $U \to 0$). Initially we took $U = 0.3$ and $a = 1.75$ from [7], table 2, assuming periodic boundary conditions. Next our initial condition was perturbed along cyclic y and the dynamic development was followed from equation (4), figure 1. The circular vortex of figure 1(left) was obtained. Its radius and velocity agree with those predicted by Jones and Roberts. The circular vortex moved forward with uniform velocity and negligible change of shape, thus confirming that it is indeed a Jones and Roberts' solution to a high degree of accuracy. We repeated the calculation for different initial conditions, always obtaining viable circular vortex solutions, see figure 2. For large R, where the tables of Jones and Roberts no longer extend, we can use the approximate formula derived by Roberts and Grant for comparison [12]:

$$U = \frac{1}{2R}[\ln(8R) - 0.615]. \tag{7}$$

See section 5 for numerical details.

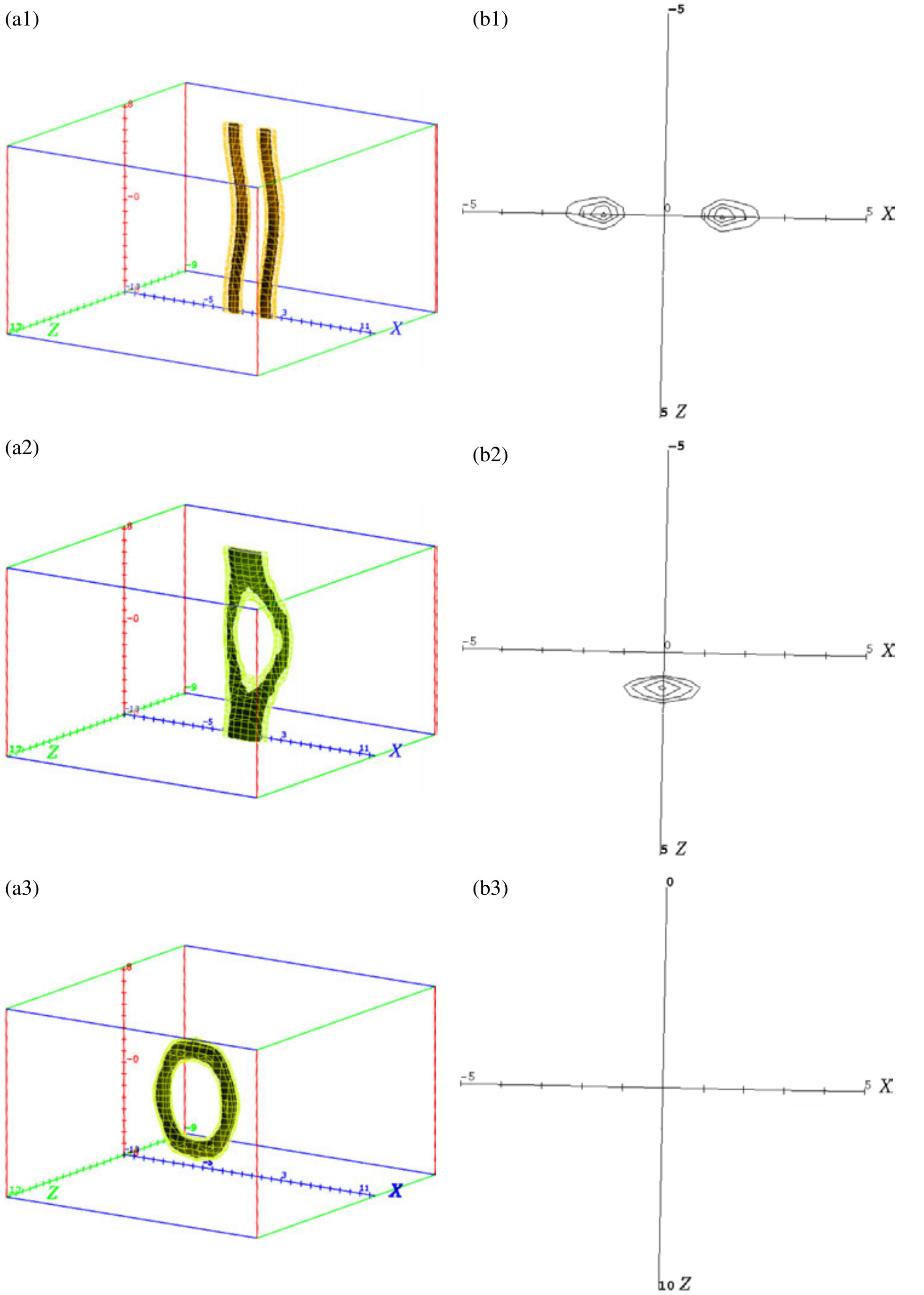

Figure 1. (Left) Three stages in a Feynman transformation of two perturbed line vortices into a circular vortex in a BEC, as follows from the NLS equation. Densities on the vortex axes are zero, and on the two indicated surfaces are 0.935 and 0.97. Lengths are in units of the healing length. The times of the three frames are 0, 7.1 and 169. (Right) Two-dimensional blow-ups of the top reconnection region, 14 healing lengths up from the origin. The slice has been chosen so as to include the crossover point. The density of the outermost contour is 0.1. Times are 0, 13 and 169.

(This figure is in colour only in the electronic version)

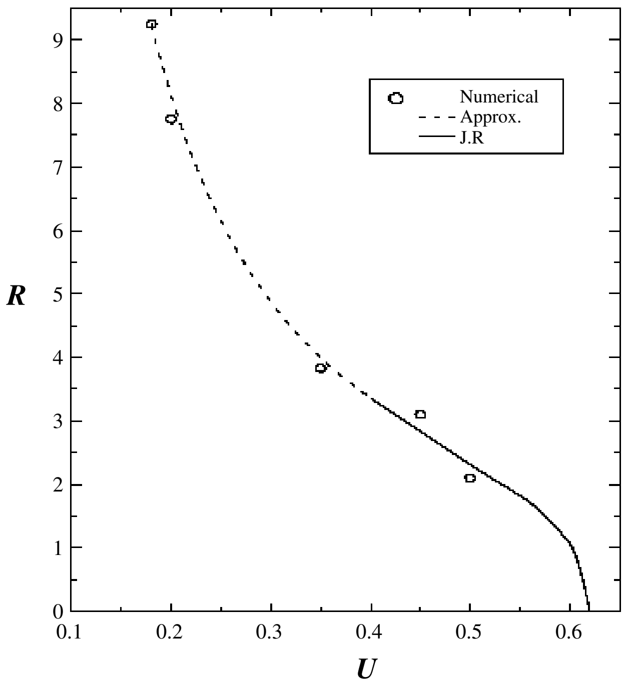

Figure 2. Our numerically obtained circular vortices (circles) as compared to those of Jones and Roberts [7], table 1 (full curve) and those of Roberts and Grant for large R [12], our equation (7) (broken curve) in R, U space.

4. Conclusions of the simulations

The two curves merge extremely smoothly in figure 2 and the circles corresponding to numerical results all lie on or very near one or the other. Thus, Feynman's hypothesis is confirmed, completing the tentative steps of [6, 8]. Of course, this confirmation is only as conclusive as the NLS model is for a Bose gas, but which is nevertheless known to be a very successful model. With the above reservation, the experimentally found abundance of circular vortices in superfluid He II is now explained theoretically. The proximity at two points of two opposite line vortices in so tangled a web is quite commonplace [2]. Both vortex lines and circles have also been created in pure BECs [13].

The generation of vortex rings due to the helical instability of vortex lines is also of primary interest in superconductivity theory [14, 15]. Perhaps our experience could be useful there, though unfortunately the equations are more complicated (in the Ginzburg–Landau model, the vector potential **A** appears in an extension of (1). An additional vector equation relates **A** and ψ).

5. Numerics

The algorithm was leapfrog as improved by Fornberg and Whitham [16]. One can show that this algorithm is marginally stable for sufficiently small time steps [17]. For example, our time step when producing figures 1 and 2 was 1.4×10^{-2}, a factor of five smaller than the critical value for stability following from the theory of [17]. The number of grid points was 36 for x, 32 for y and 128 for z. So, to check the sensitivity to the number of grid points independently, we increased it threefold and next halved it in a further simulation. In neither case did we observe any significant difference as compared to the original simulation. We never reduced to a number of grid points that would cause a difference in the dynamics.

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